Tail estimates for homogenization theorems in random media

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July 3, 2006

Abstract

It is known that a random walk on $\mathbb{Z}^d$ among i.i.d. uniformly elliptic random bond conductances verifies a central limit theorem. It is also known that approximations of the covariance matrix can be obtained by considering periodic environments. Here we estimate the speed of convergence of this homogenization result. We obtain similar estimates for finite volume approximations of the effective conductance and of the lowest Dirichlet eigenvalue. A lower bound is also given for the variance of the Green function of a random walk in a random non-negative potential.

Keywords : periodic approximation, fluctuations, effective diffusion matrix, effective conductance, non-uniform ellipticity

Subject classification : 60K37, 35B27, 82B44

1 Introduction

Consider a reversible random walk on $\mathbb{Z}^d$, $d \geq 1$, with probability transitions given by independent bond conductances, see (1). It is known from the work of Sidoravicius and Sznitman [22] that if the conductances are uniformly elliptic then a functional central limit theorem holds. Let $\mathcal{D}_0$ be the diffusion matrix.

A survey of various approximations and bounds for $\mathcal{D}_0$ can be found in [13, chap. 5-7] and [14, chap. 6-7] for this model and for related ones. As Owhadi [19] showed for the jump process in a stationary random environment, $\mathcal{D}_0$ can be approximated by the diffusion matrix of random walks in environments with bond conductances that are N-periodic.

Bourgeat and Piatnitski [5], using results from Yurinskij [27], showed that, for a similar model, under a mixing condition, the diffusion matrices converge to the homogenized matrix $\mathcal{D}_0$ faster than $CN^{-\alpha}$ where C is a constant and α is a positive exponent which depends on the dimension and the ellipticity constant.

The goal of this paper, is to obtain tail estimates for the fluctuations about the mean of the finite volume periodic approximations and to improve the estimates given in Caputo and Ioffe [6, (1.3)].

In order to do so we apply a martingale method developed by Kesten in [16] for first passage percolation models. This method also applies to other models where there is homogenization and when some regularity results are available. In all three situations that will be considered, the quantities involved are similar to first-passage times in that they can be expressed as solutions of a variational problem. It is this aspect that will be exploited.

In the second situation, tail estimates are given for the effective conductances of a cube. The estimates are interesting for dimensions $d \geq 3$. Fontes and Mathieu [11] considered random walks on $\mathbb{Z}^d$ with non-uniformly elliptic conductances. In particular, they obtained estimates on the decay of the mean return probability. Under similar conditions, we can prove estimates of the effective conductance of a cube. A lower bound on the variance for some distributions of the conductances was given by Wehr [25].

In the third situation, tail estimates are obtained for the spectral gap of a random walk on cubes in $\mathbb{Z}^d$, $d \geq 3$, with Dirichlet boundary conditions.

Kesten's martingale method was also used by Zerner [28] to study a random walk in a non-negative random potential. By this method, Zerner obtained upper bounds on the variance of the Green function. We end this paper with a short calculation leading to a lower bound.

Here are some notations that will be used throughout this article. On $\mathbb{R}^d$, $d \geq 1$, the ℓ_1-distance, the Euclidean distance and the ℓ_∞-distance will respectively be denoted by $|\cdot|_1$, $|\cdot|$ and $|\cdot|_\infty$. $\eta \in \mathbb{R}^d$ will be considered as a column vector and its transpose will be denoted by η' so that $\eta^2 = \mathrm{tr}(\eta\eta')$. We say that two vertices of $x, y \in \mathbb{Z}^d$ are neighbours, and we will write $x \sim y$, if $|x - y| = 1$. For $u : \mathbb{Z}^d \to \mathbb{R}^d$, let $\|u\|_\infty = \sup_{x\in\mathbb{Z}^d} |u(x)|_\infty$. For $m < n \in \mathbb{N}$, $[\![m, n]\!] = [m, n] \cap \mathbb{N}$. With this notation, for an integer $N \geq 1$, Q_N will be the cube in $\mathbb{Z}^d$ defined by $Q_N = [\![1, N]\!]^d$, $\overline{Q}_N = [\![0, N + 1]\!]^d$ and its boundary is $\partial Q_N = \overline{Q}_N \setminus Q_N$.

2 The stationary environment

Let $a(x, y, \omega)$, $x, y \in \mathbb{Z}^d$, $x \sim y$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, I\!\!P)$ such that $a(x, y, \omega) = a(y, x, \omega)$ for all $x \sim y$. The random variable $a(x, y, \omega)$ can be interpreted as the electric or thermic conductance of the edge joining x and y.

We will assume that this sequence is stationary. That is, there is a group of measure preserving transformations $(T_x; x \in \mathbb{Z}^d)$ acting on $(\Omega, \mathcal{F}, I\!\!P)$ such that for all $x \sim y$ and $z \in \mathbb{Z}^d$, $a(x + z, y + z, \omega) = a(x, y, T_z\omega)$. The expectation with respect to $I\!\!P$ will be denoted by $I\!\!E$ or by $\langle\cdot\rangle$.

Let $\mathcal{L}^d$ be the set of edges in $\mathbb{Z}^d$. In an environment ω, the conductance of an edge $e \in \mathcal{L}^d$ with endpoints $x \sim y$ will be denoted by $a(x, y, \omega)$ or by $a(e, \omega)$.

For most results, we will also assume that the conductances are uniformly elliptic: there is a constant $\kappa \geq 1$, called the ellipticity constant, such that for all $x, y \in \mathbb{Z}^d$, $x \sim y$, and $I\!\!P$-a.s.,

$$\kappa^{-1} \leq a(x, y, \omega) \leq \kappa.$$

Given an environment $\omega \in \Omega$, let $(X_n; n \geq 0)$ be the reversible random walk on $\mathbb{Z}^d$ with transition probabilities given by

$$p(x, y, \omega) = a(x, y, \omega)/a(x, \omega), \quad x \sim y, \tag{1}$$

where $a(x, \omega) = \sum_{y \sim x} a(x, y, \omega)$ is a stationary measure. These transition probabilities induce a probability $P_{z,\omega}$ on the paths of the random walk starting at $z \in \mathbb{Z}^d$. The corresponding expectation will be denoted by $E_{z,\omega}$.

The following proposition can be found under various forms in [4], [14], [17], [18] among others. It can be shown using Lax-Milgram lemma and Weyl's decomposition. The corrector field can also be constructed directly using the resolvent of the semigroup.

Proposition 1 *Let $(a(e); e \in \mathcal{L}^d)$ be a stationary sequence of uniformly elliptic conductances. Then there is a unique function $\chi_0 : \mathbb{Z}^d \times \Omega \to \mathbb{R}^d$, called the corrector field of the random walk, which verifies*

1. *$\chi_0(x, \cdot) \in (L^2(I\!\!P))^d$ for all $x \in \mathbb{Z}^d$,*
2. *$\int_\Omega \chi_0(x, \omega) I\!\!P(d\omega) = 0$ for all $x \in \mathbb{Z}^d$,*
3. *the cocycle property : for all $x, y \in \mathbb{Z}^d$ and $I\!\!P$-a.s.*
$$\chi_0(x + y, \omega) = \chi_0(x, \omega) + \chi_0(y, T_x\omega).$$
4. *the Poisson equation : for all $x \in \mathbb{Z}^d$ and $I\!\!P$-a.s.*
$$E_{x,\omega}(X_1 + \chi_0(X_1)) = x + \chi_0(x).$$

The last property shows that, $I\!\!P$-a.s., $X_n + \chi_0(X_n)$, $n \geq 0$, is a martingale under $P_{0,\omega}$. This martingale and the corrector field are carefully investigated in [22] to prove that, $I\!\!P$-a.s., the reversible random walk starting at the origin verifies a functional central limit theorem. In particular, it verifies a central limit theorem with a covariance matrix given by

$$\mathcal{D}_0 = \langle a(0) \rangle^{-1} \int \sum_\Lambda a(0, z)(z + \chi_0(z))(z + \chi_0(z))' dI\!\!P. \tag{2}$$

Note that in a stationary environment $\mathcal{D}_0$ might not be a diagonal matrix.

3 The periodic approximation

Given an environment $\omega \in \Omega$ and an integer $N \geq 1$, introduce an environment N-periodic on $\mathbb{Z}^d$, $d \geq 1$, by setting

$$\dot{a}_N(x, y, \omega) = a(\dot{x}, \dot{y}, \omega), \quad x \sim y$$

where $\dot{x}, \dot{y} \in [\![0, N]\!]^d$, $\dot{x} \sim \dot{y}$ and $\dot{x} \equiv x$, $\dot{y} \equiv y$ mod N coordinatewise.

Then consider the reversible random walk on $\mathbb{Z}^d$ with transition probabilities given by

$$\dot{p}_N(x, y, \omega) = \dot{a}_N(x, y, \omega)/\dot{a}_N(x, \omega), \quad x \sim y$$

where $\dot{a}_N(x, \omega) = \sum_{y \sim x} \dot{a}_N(x, y, \omega)$. These induce a probability $\dot{P}_{z,N,\omega}$ on the paths starting at $z \in \mathbb{Z}^d$. The corresponding expectation will be denoted by $\dot{E}_{z,N,\omega}$. We will also use the Laplacian $\dot{H}_{N,\omega}$ which is defined on the set of functions $u : \mathbb{Z}^d \to \mathbb{R}$ by $\dot{H}_{N,\omega}u(x) = u(x) - \dot{E}_{x,N,\omega}u(X_1)$.

3.1 Periodic corrector fields

As it was done for stationary environments, it is important to construct a periodic corrector field $\dot{\chi}_N : \mathbb{Z}^d \times \Omega \to \mathbb{R}^d$ for the random walk $(X_n; n \geq 0)$ in the periodic environment so that $I\!P$-a.s.

$$X_n + \dot{\chi}_N(X_n), \qquad n \geq 0,$$

is a martingale with respect to $\dot{P}_{0,N,\omega}$.

Therefore, $I\!P$-a.s., $\dot{\chi}_N$ must verify the equations $\dot{E}_{x,N}(X_1 + \dot{\chi}_N(X_1)) = x + \dot{\chi}_N(x)$ for all $x \in \mathbb{Z}^d$, or equivalently,

$$\dot{H}_N \dot{\chi}_N(x) = \dot{d}_N(x), \quad \text{for all } x \in \mathbb{Z}^d, \tag{3}$$

where $\dot{d}_N(x) = \dot{E}_{x,N}(X_1) - x$ is the drift of the walk. Note that each coordinate of $\dot{d}_N$ is N-periodic.

The vector space of N-periodic functions on $\mathbb{Z}^d$ can be identified with

$$\dot{\mathcal{H}}_N = \{u : \overline{Q}_N \to \mathbb{R} \ ; \ u(x) = u(y) \ \forall x, y \in \overline{Q}_N \text{ such that } \ x \equiv y \text{ mod } N\}.$$

Note that if $x \equiv y$ mod N then x or $y \in \partial Q_N$. Each function $v : Q_N \to \mathbb{R}$ has a unique extension to ∂Q_N which belongs to $\dot{\mathcal{H}}_N$. For $u \in \dot{\mathcal{H}}_N$, define $\dot{H}_N u$ on ∂Q_N so that it belongs to $\dot{\mathcal{H}}_N$. Then $\dot{H}_N : \dot{\mathcal{H}}_N \to \dot{\mathcal{H}}_N$ is a bounded linear operator.

For two functions $u, v : \overline{Q}_N \to \mathbb{R}$, define the norm, the scalar product, and the Dirichlet form respectively by $\|u\|^p_{p,\dot{N}} = \sum\limits_{x \in Q_N} |u(x)|^p \dot{a}_N(x)$, $1 \leq p < \infty$, $(u, v)_{\dot{N}} = \sum\limits_{x \in Q_N} u(x)v(x)\dot{a}_N(x)$ and

$$\dot{\mathcal{E}}_N(u, v) \quad = \quad \sum_{x,y} \dot{a}_N(x, y)(u(x) - u(y))(v(x) - v(y))$$

where the sum is over all ordered pairs $\{x, y\}$ such that $x \in Q_N$ and $y \in [\![0, N]\!]^d$.

This expression makes sense for all functions $u, v : \mathbb{Z}^d \to \mathbb{R}$. But if both u, v are N-periodic, then the Green-Gauss formula holds

$$\dot{\mathcal{E}}_N(u, v) = (u, \dot{H}_N v)_{\dot{N}}. \tag{4}$$

Let $\dot{\mathcal{H}}_N^0(\omega) = \{u \in \dot{\mathcal{H}}_N \ ; \ (u,1)_{\dot{N}} = 0\}$. Note that $\dot{\mathcal{H}}_N^0$ depends on the environment but $\dot{\mathcal{H}}_N$ does not.

All the properties of the solutions of a Poisson equation that will be needed are gathered in the following proposition.

Proposition 2 *Let $(a(e); e \in \mathcal{L}^d)$, $d \geq 1$, be a stationary sequence of uniformly elliptic conductances. Then $I\!P$-a.s.,*

1. *$\dot{H}_N : \dot{\mathcal{H}}_N^0 \to \dot{\mathcal{H}}_N^0$ is a bounded invertible linear operator .*
2. *For $f \in \mathcal{H}_N$, $\dot{H}_N u = f$ possesses a solution $u \in \dot{\mathcal{H}}_N$ if and only if $f \in \dot{\mathcal{H}}_N^0$.*
3. *Let $f \in \dot{\mathcal{H}}_N^0$.*
   - *(a) The infimum $\inf_{\dot{\mathcal{H}}_N^0} [\dot{\mathcal{E}}_N(u,u) - 2(u,f)]$ is attained by the solution $u \in \dot{\mathcal{H}}_N^0$ of the equation $\dot{H}_N u = f$.*
   - *(b) The infimum $\gamma := \inf_{\mathcal{M}} \dot{\mathcal{E}}_N(u,u)$ where $\mathcal{M} = \{u \in \dot{\mathcal{H}}_N^0 \ ; \ (f,u)_{\dot{N}} = 1\}$ is attained by the solution $u \in \dot{\mathcal{H}}_N^0$ of the equation $\dot{H}_N u = \gamma f$.*
4. *If $f \in \dot{\mathcal{H}}_N^0$ then the unique solution $u \in \dot{\mathcal{H}}_N^0$ of $\dot{H}_N u = f$ is*
$$u = \int_0^\infty e^{-t\dot{H}_N} f dt, \quad x \in Q_N.$$
5. *There is a constant $C = C(d,\kappa) < \infty$ such that for all $N \geq 1$ and $f \in \dot{\mathcal{H}}_N^0$, $u \in \dot{\mathcal{H}}_N^0$, the solution of $\dot{H}_N u = f$, verifies the regularity estimates,*
$$\|u\|_\infty \leq CN^2 \|f\|_{2,\dot{N}} \quad \text{and} \quad \|u\|_\infty \leq CN^2 (\log N) \|f\|_\infty.$$

Proof. For all $u \in \dot{\mathcal{H}}_N^0$, $\dot{H}_N u \in \dot{\mathcal{H}}_N^0$ by the Green-Gauss formula (4).

$\dot{H}_N : \dot{\mathcal{H}}_N^0 \to \dot{\mathcal{H}}_N^0$ is invertible since if $\dot{H}_N u = 0$ then u is constant by the maximum principle.

The variational principle 3b holds for the Poisson equation on a smooth compact Riemannian manifold with $f \in \mathcal{C}^\infty$, see [12, proposition 2.6 due to Druet]. The same arguments can be used.

Suppose that f is not identically 0. Then $\mathcal{M}$ is a closed convex set which is not empty since $\|f\|_{2,N}^{-2} f \in \mathcal{M}$. Therefore the infimum, γ, is attained for some $u_0 \in \mathcal{M}$ and $\gamma > 0$ since u_0 is not constant. Then using a theorem by Lagrange, there are two constants $\alpha, \beta \in \mathbb{R}$ such that for all $x \in Q_N$,

$$2\sum_{y \sim x} (u_0(x) - u_0(y)) \dot{a}_N(x,y) - \alpha f(x) \dot{a}_N(x) - \beta \dot{a}_N(x) = 0$$

and $2\dot{H}_N u_0(x) - \alpha f(x) - \beta = 0$. Therefore, for all $\varphi \in \dot{\mathcal{H}}_N$,

$$2\dot{\mathcal{E}}_N(u_0, \varphi) = \alpha (f, \varphi)_{\dot{N}} + \beta (1, \varphi)_{\dot{N}}.$$

For $\varphi = 1$, one finds that $\beta = 0$ while for $\varphi = u_0$, one finds that $\alpha = 2\gamma$. Hence $(Hu_0, \varphi)_{\dot{N}} = \gamma(f, \varphi)_{\dot{N}}$ for all $\varphi \in \dot{\mathcal{H}}_N$.

The variational principle 3a can be proven similarly.

To prove the last two properties, the estimate of the speed of convergence to equilibrium of a Markov chain on a finite state space given in terms of the spectral gap is needed. See for instance [21, Section 2.1].

Let $\dot{K}_N(t, x, y)$, $t \geq 0$ and $x, y \in Q_N$, be the heat kernel of $e^{-t\dot{H}_N}$. Then for all $t \geq 0$ and $x, y \in Q_N$, $\dot{K}_N(t, x, y) \geq 0$ and $\sum_y \dot{K}_N(t, x, y) = 1$. In particular, for all $f \in \dot{\mathcal{H}}_N$ and $t \geq 0$,

$$\|e^{-t\dot{H}_N} f\|_\infty \leq \|f\|_\infty. \tag{5}$$

Denote the volume of the torus Q_N, the invariant probability for the random walk on Q_N and the smallest non zero eigenvalue of $\dot{H}_N$ on $\dot{\mathcal{H}}_N$ respectively by

$$\dot{a}_N(Q_N) = \sum_{x \in Q_N} \dot{a}_N(x), \quad \dot{\pi}_N(x) = \dot{a}_N(x)/\dot{a}_N(Q_N) \quad \text{and} \quad \dot{\lambda}_N.$$

Then for all $t > 0$ and $x, y \in Q_N$,

$$|\dot{K}_N(t, x, y) - \dot{\pi}_N(y)| \leq \kappa \exp\big(- t\dot{\lambda}_N\big). \tag{6}$$

Therefore, for all $t > 0$ and $f \in \dot{H}^0_N$,

$$\|e^{-t\dot{H}_N} f\|_\infty = \sup_x |\sum_y (\dot{K}_N(t, x, y) - \dot{\pi}_N(y)) f(y)| \leq \kappa \exp(-t\dot{\lambda}_N) \|f\|_{1,\dot{N}} \tag{7}$$

By the Riesz-Thorin interpolation theorem, from (5) and (7), we obtain that

$$\|e^{-t\dot{H}_N} f\|_\infty \leq \sqrt{\kappa} \exp(-t\dot{\lambda}_N/2) \|f\|_{2,\dot{N}} \tag{8}$$

This will be completed by the following lower bound on $\dot{\lambda}_N$. There exists a constant $C_1 > 0$ which depends only on the dimension and on the ellipticity constant κ such that $\mathbb{P}$ a.s. and for all $N \geq 1$,

$$N^2 \dot{\lambda}_N > C_1. \tag{9}$$

This follows from the Courant-Fischer min-max principle [21, p. 319] by comparison with the eigenvalues of the simple symmetric random walk which corresponds to the case where the conductance of every edge is 1. For Neumann boundary conditions the expressions are not as explicit but for Dirichlet and periodic boundary conditions on Q_N, the eigenvalues can be calculated explicitely much as in [23]. We find that for each $\xi \in [\![0, N[\![^d$, there is an eigenvalue for the periodic boundary conditions on Q_N, $\lambda_\xi(Q_N)$, that verifies

$$\lim_{N\to\infty} N^2 \lambda_\xi(Q_N) = \frac{\pi^2}{d} \sum_{|z|=1} (\xi \cdot z)^2 \quad \text{as} \quad N \to \infty.$$

The representation formula given in *4* follows from the spectral estimates (6) and (9). See [20] for another recent application of *4*.

The first regularity result follows from the representation formula (4) and (8) : for $f \in \dot{\mathcal{H}}_N^0$,

$$\|u\|_\infty \leq \int_0^\infty \|e^{-s\dot{H}_N} f\|_\infty ds \leq \int_0^\infty \sqrt{\kappa} e^{-s\dot{\lambda}_N/2} \|f\|_{2,\dot{N}} ds \leq CN^2 \|f\|_{2,\dot{N}}.$$

The second one follows from (5) and (7) : for $f \in \dot{\mathcal{H}}_N^0$ and $t > 0$,

$$\|u\|_\infty \leq \int_0^t \|e^{-s\dot{H}_N} f\|_\infty ds + \int_t^\infty \|e^{-s\dot{H}_N} f\|_\infty ds \leq t\|f\|_\infty + \frac{e^{-t\dot{\lambda}_N}}{\dot{\lambda}_N} \|f\|_{1,\dot{N}}$$

Use $\|f\|_{1,\dot{N}} \leq 2d\kappa N^d \|f\|_\infty$ and let $t = \dfrac{d}{C_1} N^2 \log N$. □

For a function $u : \overline{Q}_N \to \mathbb{R}^d$, define $\dot{H}_N u$ in Q_N by applying it coordinate-wise.

Let $g : \mathbb{Z}^d \to \mathbb{Z}^d$ be the function defined by $g(x) = x$.

Corollary 1 *Let $(a(e); e \in \mathcal{L}^d)$, $d \geq 1$, be a stationary sequence of uniformly elliptic conductances. Then for all $N \geq 1$, there is a unique function $\dot{\chi}_N : \overline{Q}_N \times \Omega \to \mathbb{R}^d$ such that $I\!\!P$-a.s., in each coordinate, it is in $\dot{\mathcal{H}}_N^0(\omega)$ and*

$$\dot{H}_N \dot{\chi}_N = -\dot{H}_N g, \quad \text{on } Q_N. \tag{10}$$

Moreover, there is a constant $C = C(d, \kappa)$ such that

$$\|\dot{\chi}_N\|_\infty \leq CN^2 \log N. \tag{11}$$

Proof. Note that for each coordinate of $-\dot{H}_N g$ belongs to $\dot{\mathcal{H}}_N^0$:

Indeed $\dot{H}_N g$ is N-periodic and

$$\begin{aligned} \sum_{x \in Q_N} \dot{H}_N g(x) \dot{a}_N(x) &= \sum_x \sum_{y \sim x} \dot{a}_N(x) \dot{p}_N(x,y)(g(x) - g(y)) \\ &= \sum_x \sum_{y \sim x} \dot{a}_N(x,y)(x - y) = 0. \end{aligned}$$

Then use proposition 2 for the function $f = -\dot{H}_N g$. The regularity estimate (11) follows from property 5 since $\|f\|_\infty \leq 1$. □

The next step is to express the covariance matrix of the walk in a periodic environment in terms of $\dot{\chi}_N$. By (3), $M_n = X_n + \dot{\chi}_N(X_n)$, $n \geq 0$ is a martingale with uniformly bounded increments : $Z_n = M_n - M_{n-1}$.

Let $h(x) = \dot{E}_{x,N}(Z_1 Z_1')$. Since $h \in \dot{\mathcal{H}}_N$, by the ergodic theorem for a Markov chain on Q_N, $\dot{P}_{0,N}$ a.s.,

$$\frac{1}{n}\sum_1^n \dot{E}_{0,N}(Z_j Z_j' \mid X_{j-1}) \quad = \quad \frac{1}{n}\sum_1^n h(X_{j-1}) \to \sum_{Q_N} \dot{\pi}_N(x) h(x) \text{ as } n \to \infty.$$

Then by the martingale central limit theorem (see [10, (7.4) chap. 7]), $\frac{1}{\sqrt{n}} M_n$ converges to a Gaussian law. Hence $\frac{1}{\sqrt{n}} X_n$ also converges to a Gaussian with the same covariance matrix which is given by

$$\dot{\mathcal{D}}_N = \sum_{Q_N} \dot{\pi}_N(x) h(x) \tag{12}$$

$$= \dot{a}_N(Q_N)^{-1} \sum_{x \sim y} \dot{a}_N(x,y)(\dot{v}_N(y) - \dot{v}_N(x))(\dot{v}_N(y) - \dot{v}_N(x))'.$$

where $\dot{v}_N(x) = x + \dot{\chi}_N(x)$.

For uniformly elliptic, stationary and ergodic conductances, Owhadi [19, theorem 4.1] showed that for the jump process $\dot{\mathcal{D}}_N$, the effective diffusion matrix in the periodic environment, converges to the homogenized effective diffusion matrix $\mathcal{D}_0$. And since the jump process and the random walk on $\mathbb{Z}^d$ have the same diffusion matrix, the convergence theorem holds: $I\!\!P$-a.s. as $N \to \infty$,

$$\dot{\mathcal{D}}_N \to \mathcal{D}_0. \tag{13}$$

It is shown in [19] using the continuity of Weyl's decomposition and in [5] for a diffusion with random coefficients. They are both illustrations of the principle of periodic localization [14, p. 155]. At the end of this section, a terse proof by homogenization is given.

To write $\dot{\mathcal{D}}_N$ in terms of the Dirichlet form on $\dot{\mathcal{H}}_N$, we will extend the definition of $\dot{\mathcal{E}}_N$ to $\mathbb{R}^d$-valued functions so that the expression of $\dot{\mathcal{D}}_N$ given in (12) becomes

$$\dot{\mathcal{D}}_N = \dot{a}_N(Q_N)^{-1} \dot{\mathcal{E}}_N(\dot{v}_N, \dot{v}_N)$$

where $\dot{v}_N = g + \dot{\chi}_N$.

For two functions $u, v : \overline{Q}_N \to \mathbb{R}^d$, define

$$\begin{aligned} (u,v)_{\dot{N}} &= \sum_{x \in Q_N} u(x) v(x)' \dot{a}_N(x), \\ \text{and} \quad \dot{\mathcal{E}}_N(u,v) &= \sum_{x,y} \dot{a}_N(x,y)(u(x) - u(y))(v(x) - v(y))' \end{aligned}$$

where the sum is over all ordered pairs $\{x, y\}$ such that $x \in Q_N$ and $y \in [\![0, N]\!]^d$.

Coordinatewise, the periodic corrector fields are the solutions of variational problems. Indeed, from the variational formula in 3a of proposition 2, we have that $I\!P$ a.-s. and for all $N \geq 1$,

$$\begin{aligned}\inf\{\operatorname{tr}\dot{\mathcal{E}}_N(g+u \quad , \quad & g+u); u \in (\dot{\mathcal{H}}_N^0)^d\} \\ = \quad & \operatorname{tr}\dot{\mathcal{E}}_N(g,g) + \inf\{\operatorname{tr}\dot{\mathcal{E}}_N(u,u) - 2\operatorname{tr}(f,u); u \in (\dot{\mathcal{H}}_N^0)^d\} \\ = \quad & \operatorname{tr}\dot{\mathcal{E}}_N(\dot{v}_N, \dot{v}_N)\end{aligned}$$

where $g(x) = x$ and $f = -\dot{H}_N g$ as in corollary 1.

In particular, since

$$\operatorname{tr}\dot{\mathcal{E}}_N(\dot{\chi}_N, \dot{\chi}_N) \leq 2\operatorname{tr}\dot{\mathcal{E}}_N(\dot{v}_N, \dot{v}_N) + 2\operatorname{tr}\dot{\mathcal{E}}_N(g,g) \leq 4\operatorname{tr}\dot{\mathcal{E}}_N(g,g),$$

there is a constant $C = C(d,\kappa) < \infty$ such that $I\!P$-a.s. and for all $N \geq 1$,

$$\operatorname{tr}\dot{\mathcal{E}}_N(\dot{\chi}_N, \dot{\chi}_N) \leq CN^d. \tag{14}$$

The second variational principle, 3b of proposition 2, could be used to obtain a lower bound on $\operatorname{tr}\dot{\mathcal{E}}_N(\chi_N, \chi_N)$.

3.2 Further regularity results

In the following proposition, we improve the estimate given in (11) for dimensions $2 \leq d \leq 4$.

Proposition 3 *Let $(a(e); e \in \mathcal{L}^d)$, $d \geq 2$, be a stationary sequence of uniformly elliptic conductances. Then there is a constant $C = C(d,\kappa) < \infty$, such that for all $N \geq 1$*

$$\|\dot{\chi}_N\|_\infty \leq \begin{cases} CN^{d/2} & \text{for } d \geq 3 \\ CN(\log N)^{1/2} & \text{for } d = 2 \end{cases} \tag{15}$$

Proof. For $\eta \in \mathbb{R}^d$, let z_0 and z_1 be two vertices of $\mathbb{Z}^d$ such that

$$\eta \cdot \dot{\chi}_N(z_0) = \min_{x \in \mathbb{Z}^d} \eta \cdot \dot{\chi}_N(x) \quad \text{and} \quad \eta \cdot \dot{\chi}_N(z_1) = \max_{x \in \mathbb{Z}^d} \eta \cdot \dot{\chi}_N(x).$$

Since $\dot{\chi}_N$ is N-periodic, we can assume that $|z_0 - z_1|_\infty \geq N$.

Let $z_0 = x_0, x_1, \ldots, x_{n-1}, x_n = z_1$ be a path from z_0 to z_1 such that $1 \leq n \leq dN$.

For $i = 0$ or 1, let $\overline{w}_i(y) = \left(\dfrac{|z_1 - z_0|_\infty}{1 + |y - z_i|_\infty}\right)^{d-1}$ and

$$\mathcal{P}_i = \{y \in \mathbb{Z}^d; |y - z_i|_\infty \leq \frac{1}{2}|z_1 - z_0|_\infty\}.$$

For $\mathcal{C}$, a given set of finite paths in $\mathbb{Z}^d$, let $w(y) = \operatorname{card}\{\gamma \in \mathcal{C}; y \in \gamma\}$.

Then by [3, lemma 2, p.26], there is a constant $C < \infty$, which depends only on the dimension d, and there is a set of paths $\mathcal{C}$ from z_0 to z_1 such that

$$\operatorname{card} \mathcal{C} = |z_1 - z_0|_\infty^{d-1}$$

and such that for all $y \in \mathbb{Z}^d$,

$$w(y) \le C\, \overline{w}_i(y) \;\text{ if } y \in \mathcal{P}_i \text{ and } w(y) = 0 \text{ otherwise.} \tag{16}$$

For each path of $\mathcal{C}$, $z_0 = x_0, x_1 \ldots, x_{n-1}, x_n = z_1$ and for all $x \in \mathbb{Z}^d$,

$$|\eta \cdot \dot{\chi}_N(x)|_\infty \le |\eta \cdot \dot{\chi}_N(z_1) - \eta \cdot \dot{\chi}_N(z_0)|_\infty \le |\eta|_\infty \sum_{j=1}^{n} |\dot{\chi}_N(x_j) - \dot{\chi}_N(x_{j-1})|_\infty$$

Since this holds for all paths in $\mathcal{C}$, it also holds for the arithmetic average over the paths of $\mathcal{C}$. Therefore,

$$|\eta \cdot \dot{\chi}_N(x)|_\infty \le \frac{|\eta|_\infty}{|z_1 - z_0|_\infty^{d-1}} \sum_{y \in \mathcal{P}_0 \cup \mathcal{P}_1} w(y) h_N(y)$$

where $h_N(y) = \sum_{z \sim y} |\dot{\chi}_N(y) - \dot{\chi}_N(z)|_\infty$.

By (14) and (16), $|z_1 - z_0|_\infty^{1-d} \sum_{y \in \mathcal{P}_1} w(y) h_N(y)$

$$\begin{aligned}
&\le C |z_1 - z_0|_\infty^{1-d} \sum_{y \in \mathcal{P}_1} \overline{w}_1(y) h_N(y) \\
&\le \frac{C}{N^{d-1}} \left(\sum_{y \in \mathcal{P}_1} \overline{w}_1^2(y) \right)^{1/2} \left(\sum_{y \in \mathcal{P}_1} h_N^2(y) \right)^{1/2} \\
&\le \frac{C}{N^{d-1}} \left(\int_1^N \left(\frac{N}{r} \right)^{2(d-1)} r^{d-1} dr \right)^{1/2} \left(\operatorname{tr} \dot{\mathcal{E}}_N(\dot{\chi}_N, \dot{\chi}_N) \right)^{1/2} \\
&\le C N^{d/2} \left(\int_1^N r^{1-d} dr \right)^{1/2}
\end{aligned}$$

where the constant C now depends on κ and d.

And similarly for the sum over $\mathcal{P}_0$. □

In the next section the L^∞-estimates (11) and (15), will be combined with the following Hölder regularity result shown in [9, prop. 6.2] by J. Moser's iteration method for reversible random walks on infinite connected locally finite graphs with uniformly elliptic conductances :

Let $(a(e); e \in \mathcal{L}^d)$, $d \geq 1$, be a (non-random) sequence of uniformly elliptic conductances. Then there are constants $\alpha > 0$ and $C < \infty$, which depend only on the dimension and on the ellipticity constant, such that if for $N \geq 1$, $u : \overline{Q}_{2N} \to \mathbb{R}$ verifies $\dot{H}_{2N} u = 0$ in Q_{2N}, then for all $x, y \in Q_N$,

$$|u(x) - u(y)| < C \left(\frac{|x-y|_\infty}{N} \right)^\alpha \max_{Q_{2N}} |u|_\infty. \tag{17}$$

3.3 A proof of (13) by homogenization

This is analogous to the homogenization results of [14, chapters 7 - 9]. An appropriate framework for random walks is described in [2, section 10]. The convergence of the diffusion matrices in periodic environments is another illustration of these ideas.

The diffusion matrix is related to the matrix $\mathcal{A}_0$ defined in [2, section 5] by homogenization. In fact, $\mathcal{D}_0 = 2\langle a(0)\rangle^{-1}\mathcal{A}_0$. To see this, it suffices to note that for $\eta \in \mathbb{R}^d$, the function $f^\eta : \Omega \to \mathbb{R}^d$ defined by

$$f_i^\eta = \eta \cdot \chi_0(z_i), \quad 1 \leq i \leq d$$

where $\{z_i; 1 \leq i \leq d\}$ is the canonical basis of $\mathbb{R}^d$, is a solution of the auxiliary problem [2, equation (15)].

By (9) and (14), $I\!\!P$-a.s.

$$\|\dot{\chi}_N\|^2_{\dot{N}} \leq C \frac{1}{\dot{\lambda}_N} \operatorname{tr} \dot{\mathcal{E}}_N(\dot{\chi}_N, \dot{\chi}_N) \leq C N^{d+2}.$$

We have the following estimates in terms of the norm $\|u\|^2_{L^2(Q_N)} = \sum_{x \in Q_N} |u(x)|^2$: There is a constant $C < \infty$ such that for all $N \geq 1$,

$$\|g\|^2_{L^2(Q_N)} \leq N^{d+2}, \quad \|\dot{v}_N\|^2_{L^2(Q_N)} \leq C N^{d+2} \quad \text{and} \quad \operatorname{tr} \dot{\mathcal{E}}_N(\dot{v}_N, \dot{v}_N) \leq C N^d,$$

where $\dot{v}_N = \dot{\chi}_N + g$.

For $\varphi \in \mathcal{C}(\overline{Q})$ or $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$, let $\hat{\varphi}_N : Q_N \to \mathbb{R}$ be the function defined by

$$\hat{\varphi}_N(x) = \varphi(x/N), \quad x \in Q_N.$$

By the diagonalization process, Riesz representation theorem and [2, lemma 5], there is a subsequence $(N_k; k \geq 1)$ and a function $q \in \mathcal{H}^1(Q)^d$ such that for all $\varphi \in (\mathcal{C}(\overline{Q}))^d$ and $f \in \{1, a(0, z_i), 1 \leq i \leq d\}$ as $N \to \infty$ along the subsequence (N_k),

$$N^{-d-1} \sum_x \dot{v}_N(x) \cdot \hat{\varphi}_N(x) f(x) \quad \to \quad \langle f \rangle \int_Q q(s) \cdot \varphi(s) ds \tag{18}$$

and for all $\varphi \in (\mathcal{C}^\infty(\mathbb{R}^d))^d$,

$$N^{-d+1}\frac{1}{2}\dot{\mathcal{E}}_N(\dot{v}_N, \hat{\varphi}_N) \quad \to \quad \int_Q (\nabla\varphi)'\mathcal{A}_0\nabla q ds \tag{19}$$

where $\nabla\varphi$ is the $d \times d$ matrix $(\nabla\varphi)_{i,j} = \frac{\partial}{\partial s_j}\varphi_i$, $1 \le i, j \le d$ and similarly for ∇q. By (18),

$$\begin{aligned} N^{-d-1}\sum(\dot{\chi}_N + g)\cdot\hat{\varphi}_N &= N^{-d-1}\sum\dot{\chi}_N\cdot\hat{\varphi}_N + N^{-d}\sum\frac{x}{N}\cdot\hat{\varphi}_N \\ &\to 0 + \int_Q s\cdot\varphi(s)ds. \end{aligned}$$

Therefore $\int_Q(q-g)ds = 0$ and $\tilde{\chi}_0 \overset{\text{def}}{=} q - g$ is 1-periodic, that is $\tilde{\chi}_0 \in \dot{\mathcal{H}}^1(Q)$ and $\int_Q \tilde{\chi}_0 ds = 0$.

By (4), $\dot{\mathcal{E}}_N(\dot{v}_N, \hat{\varphi}_N) = (\dot{H}_N g, \hat{\varphi}_N)_{\dot{N}} - (\dot{H}_N g, \hat{\varphi}_N)_{\dot{N}} = 0$.

Hence by (19), $-\operatorname{div}\mathcal{A}_0\nabla(\tilde{\chi}_0 + g) = 0$. Then $\tilde{\chi}_0$ is the unique solution of the Poisson problem. Therefore there is convergence in (18) and (19) for $N \to \infty$ and in particular, $I\!\!P$-a.s.

$$\begin{aligned} \dot{\mathcal{D}}_N &= \dot{a}_N(Q_N)^{-1}\dot{\mathcal{E}}_N(\dot{v}_N, \dot{v}_N) \\ &= \dot{a}_N(Q_N)^{-1}\dot{\mathcal{E}}_N(\dot{v}_N, g) \\ &\to 2\langle a\rangle^{-1}\int_Q(\nabla g)'\mathcal{A}_0\nabla q ds \\ &= 2\langle a\rangle^{-1}\int_Q(\nabla g)'\mathcal{A}_0(\nabla g + \nabla\tilde{\chi}_0)ds \\ &= 2\langle a\rangle^{-1}\mathcal{A}_0 = \mathcal{D}_0. \end{aligned}$$

□

4 Upper bounds on tail estimates

4.1 Martingale estimates and further notations

The tail estimates and the corresponding upper bounds on the variance will be obtained by the method of bounded martingale differences developed by Kesten for first-passage percolation models [16].

When the conductances are assumed to be uniformly elliptic, a stronger version of Kesten's martingale inequality can be used. The same proof applies with some simplifications. In particular, [16, Step (iii)] is not needed. However the full generality of Kesten's martingale inequalities will be used when we consider conductances that are positive and bounded above but not necessarily uniformly elliptic. They are given in the second version.

Martingale estimates I.

Let $(M_n; n \geq 0)$ be a martingale on a probability space $(\Omega, \mathcal{F}, I\!P)$.
Let $\Delta_k = M_k - M_{k-1}$, $k \geq 1$. If there are positive random variables U_k (not necessarily $\mathcal{F}_k$-measurable) such that for some constant $B_0 < \infty$

$$I\!E(\Delta_k^2 | \mathcal{F}_{k-1}) \leq I\!E(U_k | \mathcal{F}_{k-1}) \text{ for all } k \geq 1 \quad \text{and} \quad \sum_1^\infty U_k \leq B_0, \tag{20}$$

then $M_n \to M_\infty$ in L^2 and a.s., and $I\!E|M_\infty - M_0|^2 \leq B_0$.

Moreover, if there is a constant $B_1 < \infty$ such that for all $k \geq 1$,

$$|\Delta_k| \leq B_1, \tag{21}$$

then for all $t > 0$,

$$I\!P\left(|M_\infty - M_0| > t\right) \leq 4 \exp\left(-\frac{t}{4\sqrt{B}}\right)$$

where $B = \max\{B_0, eB_1^2\}$.

Martingale estimates II.

Let $(M_n; n \geq 0)$ be a martingale on a probability space $(\Omega, \mathcal{F}, I\!P)$.
Let $\Delta_k = M_k - M_{k-1}$, $k \geq 1$. If there is a constant $B_1 < \infty$ such that for all $k \geq 1$,

$$|\Delta_k| \leq B_1, \tag{22}$$

if for some random variables $U_k \geq 0$ (not necessarily $\mathcal{F}_k$-measurable)

$$I\!E(\Delta_k^2 | \mathcal{F}_{k-1}) \leq I\!E(U_k | \mathcal{F}_{k-1}) \text{ for all } k \geq 1, \tag{23}$$

and if there are constants $0 < C_1, C_2 < \infty$ and $s_0 \geq e^2 B_1^2$ such that,

$$I\!P\Big(\sum_k U_k > s\Big) \leq C_1 \exp(-C_2 s^2), \text{ for all } s \geq s_0, \tag{24}$$

then there are universal constants c_1 and c_2 which do not depend on B_1, s_0, C_1, C_2 nor on the distribution of (M_k) and (U_k) such that for all $s > 0$,

$$I\!P\left(|M_\infty - M_0| > s\right) \leq c_1 \left(1 + C_1 + \frac{C_1}{s_0^2 C_2}\right) \exp\left(-c_2 \frac{s}{s_0^{1/2} + (s/(s_0 C_2))^{1/3}}\right)$$

In the calculations to follow, we will use the following lighter notations. In an environment ω, the conductance of an edge e with endpoints $x \sim y$ is denoted by $a(e, \omega)$ or $a(x, y, \omega)$. Similarly, for a function v which is defined for x and y, the endpoints of e, the difference $v(x, \omega) - v(y, \omega)$ will be denoted, up to a sign, by $v(e, \omega)$.

Let $\mathcal{L}^d$ be the set of edges in $\mathbb{Z}^d$. Whenever we assume that the conductances are independent, identically distributed and bounded by κ, we will also assume

that $I\!P$ is a product measure on $\Omega = [0,\kappa]^{\mathcal{L}^d}$ and $a(e,\cdot)$ are the coordinate functions.

Let $\{e_k; k \geq 1\}$ be a fixed ordering of $\mathcal{L}^d$ and let $\mathcal{F}_k$, $k \geq 1$, be the σ-algebra generated by $\{a(e_j,\cdot); 1 \leq j \leq k\}$. Then for an integrable random variable $h : \Omega \to \mathbb{R}$,

$$I\!E(h|\mathcal{F}_k) = I\!E_\sigma h([\omega,\sigma]_k)$$

where $[\omega,\sigma]_k \in \Omega$ agrees with ω for the first k coordinates and with σ for all the other coordinates and $I\!E_\sigma$ denotes the integration with respect to $dI\!P(\sigma)$.

4.2 Approximations by a periodic environment

In this section, we improve the tail estimates given in [6] by using the regularity results from sections 3.1 and 3.2 and the martingale estimates I given in section 4.1 above. Note that the calculations following the inequality [6, (4.5)] hold only for some laws $I\!P$, like discrete laws on a finite subset of $\mathbb{R}_+$. Denote the entries of $\dot{D}_N$ by $\dot{\mathcal{D}}_N^{ij}$, $1 \leq i,j \leq d$.

Theorem 1 *Let $(a(e); e \in \mathcal{L}^d)$ be a sequence of i.i.d. uniformly elliptic conductances with κ as the ellipticity constant. Then there is a constant C, $0 < C < \infty$, which depends only on the dimension and on the ellipticity constant κ such that, for all $t > 0$ and $N \geq 1$,*

$$\max_{i,j} I\!P(|\dot{\mathcal{D}}_N^{ij} - I\!E\dot{\mathcal{D}}_N^{ij}| \geq tN^{-\nu(d)}) \leq 4\exp(-Ct)\,, \tag{25}$$

$$\max_{i,j} \mathbf{V}\mathrm{ar}(\dot{\mathcal{D}}_N^{ij}) \leq 8C^{-2}N^{-2\nu(d)} \tag{26}$$

where $2\nu(d) = \max\{\alpha, d-4+\alpha\}$ and $\alpha > 0$ is the regularity exponent which appears in (17).

Let $\{z_i; 1 \leq i \leq d\}$ be the canonical basis of $\mathbb{R}^d$. For $1 \leq i \leq d$, let

$$\dot{f}_N = N^{-d} z_i' \dot{\mathcal{E}}_N(\dot{v}_N, \dot{v}_N) z_i$$

where

$$\dot{v}_N(x) = x + \dot{\chi}_N(x), \qquad x \in \overline{Q}_N.$$

Recall that by (13), $\dot{f}_N \to \langle a(0)\rangle z_i' \mathcal{D}_0 z_i$, $I\!P$ a.s. and in $L^1(I\!P)$ as $N \to \infty$.

Lemma 1 *Let ω and σ be two environments such that $a(e,\omega) = a(e,\sigma)$ for all edges e except maybe for $e = e_k$. Then for all $N \geq 1$,*

$$|\dot{f}_N(\omega) - \dot{f}_N(\sigma)| \leq \kappa N^{-d}(\dot{v}_N^2(e_k,\omega) + \dot{v}_N^2(e_k,\sigma))$$

Proof of lemma 1. By (10) and by 3a of proposition 2, $\dot{v}_N$ is the solution of a variational problem. Then $\dot{f}_N(\omega) - \dot{f}_N(\sigma)$

$$\begin{aligned} &= N^{-d} \sum_e a(e,\omega)(z_i \cdot \dot{v}_N(e,\omega))^2 - N^{-d} \sum_e a(e,\sigma)(z_i \cdot v_N(e,\sigma))^2 \\ &\leq N^{-d} \sum_e (\dot{a}_N(e,\omega) - \dot{a}_N(e,\sigma))(z_i \cdot \dot{v}_N(e,\sigma))^2 \\ &\leq \kappa N^{-d} \dot{v}_N^2(e_k,\sigma). \end{aligned}$$

□

Proof of theorem 1. Let $\Delta_k = \mathbb{E}(\dot{f}_N \mid \mathcal{F}_k) - \mathbb{E}(\dot{f}_N \mid \mathcal{F}_{k-1})$, $k \geq 1$.
Let $M_0 = 0$ and $M_n = \sum_1^n \Delta_k$, $n \geq 1$. Then $(M_n; n \geq 0)$ is a martingale and we will see that

$$\dot{f}_N - \mathbb{E}\dot{f}_N = \sum_1^\infty \Delta_k \qquad \text{a.s. and in } L^2.$$

We first check that $(M_n; n \geq 0)$ verifies conditions (20) and (21).

By (11), (15) and the Hölder regularity (17), there are constants β and $C < \infty$ which depend only on κ and d such that $\mathbb{P}$-a.s. and for all $N \geq 1$,

$$\sup_e \dot{v}_N^2(e) < CN^\beta$$

where $\beta = \min\{d - \alpha, 4 - \alpha\}$. By lemma 1, we see that

$$\begin{aligned} |\Delta_k| &= |\mathbb{E}(\dot{f}_N \mid \mathcal{F}_k) - \mathbb{E}(\dot{f}_N \mid \mathcal{F}_{k-1})| \\ &= |\mathbb{E}_\sigma(\dot{f}_N([\omega,\sigma]_k) - \dot{f}_N([\omega,\sigma]_{k-1}))| \\ &\leq \mathbb{E}_\sigma |\dot{f}_N([\omega,\sigma]_k) - \dot{f}_N([\omega,\sigma]_{k-1})| \\ &\leq \kappa N^{-d} \mathbb{E}_\sigma(\dot{v}_N^2(e_k,[\omega,\sigma]_{k-1}) + \dot{v}_N^2(e_k,[\omega,\sigma]_k)) \\ &\leq CN^{-d+\beta} \end{aligned}$$

Hence, (21) holds with $B_1 = CN^{-d+\beta}$. Similarly,

$$\begin{aligned} \Delta_k^2 &\leq \mathbb{E}_\sigma(|\dot{f}_N([\omega,\sigma]_k) - \dot{f}_N([\omega,\sigma]_{k-1})|^2) \\ &\leq 2\kappa^2 N^{-2d} \mathbb{E}_\sigma(\dot{v}_N^4(e_k,[\omega,\sigma]_{k-1}) + \dot{v}_N^4(e_k,[\omega,\sigma]_k)) \\ &\leq CN^{-2d+\beta} \mathbb{E}_\sigma(\dot{v}_N^2(e_k,[\omega,\sigma]_{k-1}) + \dot{v}_N^2(e_k,[\omega,\sigma]_k)). \end{aligned}$$

Let

$$U_k(\omega) = 2CN^{-2d+\beta} \dot{v}_N^2(e_k, \omega).$$

We see that $\mathbb{E}(\Delta_k^2 \mid \mathcal{F}_{k-1}) \le \mathbb{E}(U_k \mid \mathcal{F}_{k-1})$ and that (20) holds with $B_0 = 2CN^{-d+\beta}$ since

$$\begin{aligned}\sum_k U_k &= 2CN^{-2d+\beta} \sum_k \dot{v}_N^2(e_k, \omega) \\ &\le CN^{-2d+\beta} \operatorname{tr} \dot{\mathcal{E}}(\dot{v}_N, \dot{v}_N)_{N,\omega} \le CN^{-2d+\beta+d} = CN^{-d+\beta}.\end{aligned}$$

Then the martingale estimates I hold with $B = \max\{B_0, eB_1^2\} = B_0$. Hence, for all $t > 0$,

$$\mathbb{P}(|\dot{f}_N - \mathbb{E}\dot{f}_N| \ge t) \le 4 \exp\left(-CtN^{(d-\beta)/2}\right)$$

and for all $N \ge 1$, $\mathbf{V}\mathrm{ar}(\dot{f}_N) \le CN^{\beta - d}$. □

4.3 Effective conductance

In this section, we obtain similar tail estimates for the effective conductances of an increasing sequence of cubes under mixed boundary conditions and an upper bound on the variances.

It is simpler to work with these boundary conditions because instead of the L^∞ estimates (11) and (15), we use the maximum principle : if $u : \overline{Q}_N \to \mathbb{R}$ verifies $Hu = 0$ on Q_N then $\max_{\overline{Q}_N} u = \max_{\partial Q_N} u$. Furthermore, since the maximum principle does not require uniform ellipticity, it is possible to obtain good estimates under weaker conditions on the conductances. After the description of the model, we state two theorems. The first one gives some tail estimates when the conductances are uniformly elliptic while the second one is when they are not.

Consider boundary conditions which can be interpreted as maintaining a fixed potential difference between two opposite faces of $Q_N = [\![1, N]\!]^d$ while the other faces are insulated. Denote the first coordinate of $x \in \mathbb{Z}^d$ by $x(1)$. Let $\mathcal{V}_N$ be the set of real-valued functions on $\overline{Q}_N$ such that

$$\begin{aligned} &u = 0 \text{ on } \{x(1) = 0\} \cap \partial Q_N, \qquad u = N+1 \text{ on } \{x(1) = N+1\} \cap \partial Q_N \\ &\qquad\qquad \text{and} \quad u(x) = u(y) \\ &\text{for all } x \sim y, x \in \{x(1) \ne 0\} \quad \cap \quad \{x(1) \ne N+1\} \cap \partial Q_N, y \in Q_N. \end{aligned}$$

The Dirichlet form on $\mathcal{H}_N$, will be denoted by $\mathcal{E}_N$. For two functions $u, v : \overline{Q}_N \to \mathbb{R}$, it is defined by

$$\mathcal{E}_N(u, v) = \sum_{x,y} a(x,y)(u(x) - u(y))(v(x) - v(y)) \tag{27}$$

where the sum is over all ordered pairs $\{x, y\}$ such that $x \in Q_N$ and $y \in \overline{Q}_N$.

If for all edges e of $\mathbb{Z}^d$, $a(e) > 0$, then for all $N \geq 1$, there is a unique $v_N \in \mathcal{V}_N$ such that $Hv_N = 0$ in Q_N. v_N is also a solution of a variational problem : it is the unique element of $\mathcal{V}_N$ such that

$$\mathcal{E}_N(v_N, v_N) = \inf \left\{ \ \mathcal{E}_N(u, u); u \in \mathcal{V}_N \right\}.$$

We will also write $\mathcal{E}_N(v_N, v_N)_\omega$ to indicate that $\mathcal{E}_N$ and v_N are calculated with the conductances $a(e, \omega)$ given by the environment ω.

Let $f_N(\omega) = N^{-d}\mathcal{E}_N(v_N, v_N)_\omega$. It can be interpreted as the effective conductance between opposite faces of Q_N.

This model was considered by Wehr [25]. He showed that for some laws, $I\!P$, which include the exponential and the one-sided normal distributions, and assuming that $I\!E(f_N)$ is bounded below by a positive constant, then

$$\liminf_N N^d \, \mathbf{V}\mathrm{ar}(f_N) > 0.$$

If the conductances are uniformly elliptic and stationary, then $I\!P$ a.s. and in $L^1(I\!P)$, as $N \to \infty$, f_N converges to the effective conductance $f_0 = \nabla v_0 \mathcal{A}_0 \nabla v_0$ where $\mathcal{A}_0 = \langle a(0) \rangle \mathcal{D}_0$ is given in theorem 2 and v_0 is the solution of a variational problem with mixed boundary conditions. This was done in [2, section 10] by adapting the homogenization methods of [14, chapter 7].

Theorem 2 *Let $(a(e); e \in \mathcal{L}^d)$ be a sequence of i.i.d. uniformly elliptic conductances on $\mathbb{Z}^d$, $d \geq 3$, with ellipticity constant $\kappa \geq 1$. Let $C_0 = 32d\kappa^3$.*

Then for all $t \geq 0$ and $N \geq 1$,

$$I\!P(|f_N - I\!E f_N| \geq tN^{(2-d)/2}) \leq 4 \exp\left(-t/\sqrt{\kappa C_0}\right)$$

and

$$\mathbf{V}\mathrm{ar}(f_N) \leq \kappa C_0 N^{2-d}.$$

For conductances that are not necessarily uniformly elliptic, we have the following estimates. Additional properties of non-uniformly elliptic reversible random walks can be found in [11].

Theorem 3 *Let $(a(e); e \in \mathcal{L}^d)$ be a sequence of i.i.d. conductances on $\mathbb{Z}^d$ such that for some constant $1 \leq \kappa < \infty$, $0 < a(e) \leq \kappa$ for all edges e. Let $C_0 = 32d\kappa^3$.*

Then, for $d \geq 5$, $\mathbf{V}\mathrm{ar}(f_N) \leq 128d\kappa^2 N^{4-d}$ and for all $t > 0$ and $N \geq 1$,

$$I\!P(|f_N - I\!E f_N| \geq tN^{(4-d)/2}) \leq 4 \exp\left(-t/(8\kappa\sqrt{2d})\right) \tag{28}$$

If moreover, for some constants $D_0 < \infty$ and γ, $0 < \gamma < 2$,

$$I\!P(a^{-1}(e) \geq s) \leq D_0 s^{1-2/\gamma}, \quad \text{for all } s \geq 1, \tag{29}$$

then, if $d \geq 4$ or if $d = 3$ and $1/2 \leq \gamma < 1$,

$$\mathbf{V}\mathrm{ar}\, f_N \leq 16C_0(D_0 + 1)N^{2-d+\gamma}$$

and for all $0 < t < \left(\frac{C_0(D_0+1)^5}{8}\right)^{1/2} N^{(d-6+5\gamma)/2}$,

$$I\!P(|f_N - I\!E f_N| > t) \leq 11 c_1 \exp\left(-\frac{c_2}{2\sqrt{2C_0(D_0+1)}} N^{(d-2-\gamma)/2} t\right). \qquad (30)$$

where c_1 *and* c_2 *are the constants that appear in the martingale estimates II. In particular, they do not depend on* κ, d *or* N.

For $d = 3$ *and* $0 < \gamma \leq 1/2$, $\mathbf{V}\mathrm{ar}\, f_N \leq 16 C_0(D_0+1) N^{-1/2}$.

Lemma 2 *Let* ω *and* σ *be two environments such that* $a(e,\omega) = a(e,\sigma)$ *for all edges* e *except maybe for* $e = e_k$*. Then for all* $N \geq 1$,

$$|f_N(\omega) - f_N(\sigma)| \leq \kappa N^{-d}(v_N^2(e_k,\omega) + v_N^2(e_k,\sigma))$$

Proof. Since $v_N \in \mathcal{V}_N$ is the solution of a variational problem,

$$\begin{aligned} f_N(\omega) - f_N(\sigma) &= N^{-d}(\mathcal{E}_N(v_N, v_N)_\omega - \mathcal{E}_N(v_N, v_N)_\sigma) \\ &\leq N^{-d} \sum_e (a(e,\omega) - a(e,\sigma)) v_N^2(e,\sigma) \leq \kappa N^{-d} v_N^2(e_k,\sigma) \end{aligned}$$

□

Proof of theorem 2. With the notations of 4.1, let $\Delta_k = I\!E(f_N \mid \mathcal{F}_k) - I\!E(f_N \mid \mathcal{F}_{k-1})$, $M_0 = 0$ and $M_n = \sum_1^n \Delta_k$, $n \geq 1$.

To check that $(M_n; n \geq 0)$ is a martingale that verifies conditions (20) and (21), we have by lemma 2,

$$\begin{aligned} |\Delta_k| &\leq I\!E_\sigma \, |f_N([\omega,\sigma]_k) - f_N([\omega,\sigma]_{k-1})| \\ &\leq \kappa N^{-d} I\!E_\sigma v_N^2(e_k, [\omega,\sigma]_{k-1}) + \kappa N^{-d} I\!E_\sigma v_N^2(e_k, [\omega,\sigma]_k) \\ &\leq 8\kappa N^{2-d} \end{aligned}$$

since by the maximum principle, $0 \leq v_N(x) \leq N+1$, for all $x \in \overline{Q}_N$.

Then,

$$\begin{aligned} \Delta_k^2 &\leq I\!E_\sigma(|f_N([\omega,\sigma]_k) - f_N([\omega,\sigma]_{k-1})|^2) \\ &\leq 2\kappa^2 N^{-2d} I\!E_\sigma v_N^4(e_k, [\omega,\sigma]_{k-1}) + 2\kappa^2 N^{-2d} I\!E_\sigma v_N^4(e_k, [\omega,\sigma]_k) \\ &\qquad \text{and by the maximum principle,} \\ &\leq 8\kappa^2 N^{2-2d} I\!E_\sigma v_N^2(e_k, [\omega,\sigma]_{k-1}) + 8\kappa^2 N^{2-2d} I\!E_\sigma v_N^2(e_k, [\omega,\sigma]_k). \end{aligned}$$

For $k \geq 1$, let

$$U_k(\omega) = 16\kappa^2 N^{2-2d} v_N^2(e_k, \omega). \qquad (31)$$

We see that $I\!E(\Delta_k^2 \mid \mathcal{F}_{k-1}) \le I\!E(U_k \mid \mathcal{F}_{k-1})$ and by uniform ellipticity,

$$\begin{aligned}\sum_k U_k &= 16\kappa^2 N^{2-2d} \sum_k v_N^2(e_k,\omega) \\ &\le 16\kappa^3 N^{2-2d} \mathcal{E}_N(v_N, v_N)_\omega \\ &< 16\kappa^3 (2d\kappa) N^{2-2d+d} = 32 d\kappa^4 N^{2-d}.\end{aligned}$$

Hence condition (20) holds with $B_0 = 32d\kappa^4 N^{2-d}$ and condition (21) holds with $B_1 = 4\kappa N^{2-d}$. Therefore, $f_N - I\!E f_N = \sum_1^\infty \Delta_k$ a.s. and in L^2 and for $d \ge 3$, by the martingale estimates I, we have that for all $N \ge 1$ and $t \ge 0$,

$$I\!P(|f_N - I\!E f_N| \ge t) \le 4 \exp\left(-\frac{t}{4\sqrt{B}}\right)$$

where $B = \max\{B_0, eB_1^2\} = 32d\kappa^4 N^{2-d}$. Accordingly, $\mathbf{V}\mathrm{ar}(f_N) \le 32 d\kappa^4 N^{2-d}$. □

Proof of theorem 3. To obtain (28), the preceding proof can be used up to (31) where uniform ellipticity is first needed.

Let $U_k(\omega) = 16\kappa^2 N^{2-2d} v_N^2(e_k,\omega)$, $k \ge 1$.

Then simply bound $v_N^2(e_k,\omega)$ by $4N^2$ to obtain that

$$\sum_k U_k \le 128 d\kappa^2 N^{4-d}.$$

Hence the martingale estimates I hold with $B_0 = 128d\kappa^2 N^{4-d}, B_1 = 4\kappa N^{2-d}$ and $B = \max\{B_0, eB_1^2\} = B_0$.

These estimates can be improved if we assume that (29) holds for some $0 < \gamma < 2$. Starting from (31), we have that for all $N \ge 1$,

$$\begin{aligned}\sum_k v_N^2(e_k,\omega) &\le N^\gamma \sum_k a(e_k,\omega) v_N^2(e_k,\omega) + 4N^2 \sharp\{k; a(e_k,\omega) \le N^{-\gamma}\} \\ &\le 2d\kappa N^{d+\gamma} + 4N^2 \sharp\{k; a(e_k,\omega) \le N^{-\gamma}\}.\end{aligned}$$

Therefore, $\sum_k U_k \le 32 d\kappa^3 N^{2-d+\gamma} + 64\kappa^2 N^{4-2d} \sharp\{k; a(e_k,\omega) \le N^{-\gamma}\}$.

Let $C_0 = 32d\kappa^3$. Then for $t > 1$, $N \ge 1$ and $\gamma > 0$,

$$\begin{aligned}I\!P\left(\sum_k U_k > C_0 t N^{2-d+\gamma}\right) &\le I\!P\left(\sharp\{k; a(e_k,\omega) \le N^{-\gamma}\} > (t-1)N^{d-2+\gamma}\right) \\ &\le 2\exp\left(-\frac{N^d}{4}((t-1)N^{\gamma-2} - p_N)^2\right)\end{aligned}$$

by Bernstein's inequality with $p_N = I\!P(a^{-1}(e) > N^\gamma)$. By (29), we have that for all $N \geq 1$, $p_N \leq D_0 N^{\gamma-2}$. Therefore, if $t > 2(1+D_0)$ then $(t-1-D_0)^2 > t^2/4$ and

$$\begin{aligned} I\!P\left(\sum_k U_k > C_0 t N^{2-d+\gamma}\right) &\leq 2\exp\left(-\frac{N^d}{4}(t-1-D_0)^2 N^{2\gamma-4}\right) \\ &\leq 2\exp\left(-\frac{t^2}{16}N^{d-4+2\gamma}\right) \end{aligned}$$

Equivalently, for all $s \geq 2C_0(D_0+1)N^{2-d+\gamma}$,

$$I\!P(\sum_k U_k > s) \leq 2\exp\left(-\frac{s^2}{16C_0^2}N^{3d-8}\right).$$

We see that the conditions for the martingale estimates II hold with

$$B_1 = 4\kappa N^{2-d}, C_1 = 2, C_2 = \frac{1}{16C_0^2}N^{3d-8} \quad \text{and } s_0 = 2C_0(D_0+1)N^{2-d+\gamma}$$

since $C_0 \geq 8e^2\kappa^2$.

In particular, we have the tail estimates (30).

Moreover,

$$\begin{aligned} \mathbb{V}\mathrm{ar}\, f_N &\leq I\!E\sum_k U_k = \int_0^\infty I\!P(\sum_k U_k > s)ds \\ &\leq s_0 + \int_{s_0}^\infty C_1 e^{-C_2 s^2} ds. \end{aligned}$$

Hence, if $2-d+\gamma \geq 6-2d-\gamma$,

$$\mathbb{V}\mathrm{ar}\, f_N \leq s_0 + \frac{C_1}{s_0 C_2} \leq 16C_0(D_0+1)N^{2-d+\gamma}$$

while for $d = 3$ and $\gamma - 1 \leq -1/2$,

$$\mathbb{V}\mathrm{ar}\, f_N \leq s_0 + \frac{C_1}{\sqrt{C_2}} \leq 16C_0(D_0+1)N^{-1/2}.$$

□

4.4 Spectral gap with Dirichlet boundary conditions

In this last example, we obtain tail estimates for the spectral gap of the random walk on an increasing sequence of cubes under Dirichlet boundary conditions.

Let $\mathcal{H}_{N,0} = \{u : \overline{Q}_N \to \mathbb{R}\ ;\ u = 0 \text{ on } \partial Q_N\}$. If $u, v \in \mathcal{H}_{N,0}$ then $\mathcal{E}_N(u,v) = (Hu, v)_N$ where the Dirichlet form $\mathcal{E}_N$ is defined in (27).

Let $\psi_N \in \mathcal{H}_{N,0}$ be the solution of the variational problem :

$$\mathcal{E}_N(\psi_N, \psi_N) = \inf\left\{\ \mathcal{E}_N(u,u); u \in \mathcal{H}_{N,0},\ \|u\|_{2,N} = 1\right\}.$$

Then ψ_N is unique (up to a sign) and is an eigenfunction of H acting on $\mathcal{H}_{N,0}$. Let $\lambda_N > 0$ be the corresponding eigenvalue. It was shown in [2], by homogenization methods as in Kesavan [15], that $N^2\lambda_N$ converges $I\!\!P$-a.s. and in $L^1(I\!\!P)$ as $N \to \infty$ to the Dirichlet eigenvalue of a second-order elliptic operator with constant coefficients.

The L^∞ estimates of the eigenfunction is provided by the De Georgi-Nash-Moser theory (see [8, section 2.1] and [7, chapter 11]) :

Let $(a(e); e \in \mathcal{L}^d)$, $d \geq 3$, be a (non-random) sequence of uniformly elliptic conductances. Then there is a constant $C < \infty$ which depends only on the dimension and on the ellipticity constant such that if for $N \geq 1$, $\psi \in \mathcal{H}_{N,0}$ is a normalized eigenfunction of H, that is, for some $\lambda > 0$, $H\psi = \lambda\psi$ on Q_N, then for all $x \in Q_N$,

$$|\psi(x)| \leq C\lambda^{d/4}. \tag{32}$$

Theorem 4 *Let $(a(e); e \in \mathcal{L}^d)$ be a sequence of i.i.d. uniformly elliptic conductances on $\mathbb{Z}^d$, $d \geq 3$. Let*

$$f_N = N^2\lambda_N = N^2\mathcal{E}_N(\psi_N, \psi_N).$$

Then there is a constant $C < \infty$ which depends only on d and κ such that for all $t > 0$ and $N \geq 1$,

$$I\!\!P(|f_N - I\!\!E f_N| \geq tN^{(2-d)/2}) \leq 4\exp\left(-t/C\right).$$

and

$$\mathbf{V}\mathrm{ar}(f_N) \leq CN^{2-d}.$$

Lemma 3 *Let ω and σ be two environments such that $a(e,\omega) = a(e,\sigma)$ for all edges e except maybe at $e = e_k$, an edge with endpoints $x_k \sim y_k$ of $\mathbb{Z}^d$, where*

$$a(e_k,\omega) \geq a(e_k,\sigma).$$

Then for all $N \geq 1$,

$$|f_N(\omega) - f_N(\sigma)| \leq C(N^2\psi_N^2(e_k,\sigma) + \psi_N^2(x_k,\omega) + \psi_N^2(y_k,\omega)).$$

Proof. By the variational principle,

$$\begin{aligned}
\lambda_N(\omega) - \lambda_N(\sigma) &= \mathcal{E}_N(\psi_N(\omega),\psi_N(\omega))_\omega - \mathcal{E}_N(\psi_N(\sigma),\psi_N(\sigma))_\sigma \\
&\leq \gamma\mathcal{E}_N(\psi_N(\sigma),\psi_N(\sigma))_\omega - \mathcal{E}_N(\psi_N(\sigma),\psi_N(\sigma))_\sigma \\
&\quad \text{where } \gamma = \|\psi_N(\sigma)\|_{2,N,\omega}^{-2} \\
&= (\gamma - 1)\mathcal{E}_N(\psi_N(\sigma),\psi_N(\sigma))_\omega \\
&\quad + \mathcal{E}_N(\psi_N(\sigma),\psi_N(\sigma))_\omega - \mathcal{E}_N(\psi_N(\sigma),\psi_N(\sigma))_\sigma \\
&\leq \mathcal{E}_N(\psi_N(\sigma),\psi_N(\sigma))_\omega - \mathcal{E}_N(\psi_N(\sigma),\psi_N(\sigma))_\sigma \\
&\quad \text{since } \gamma \leq 1 \\
&\leq (a(e_k,\omega) - a(e_k,\sigma))\psi_N^2(e_k,\sigma) \\
&\leq \kappa\psi_N^2(e_k,\sigma).
\end{aligned}$$

Similarly, by the variational principle,

$$\begin{aligned}
\lambda_N(\sigma) - \lambda_N(\omega) &\leq \overline{\gamma}\mathcal{E}_N(\psi_N(\omega), \psi_N(\omega))_\sigma - \mathcal{E}_N(\psi_N(\omega), \psi_N(\omega))_\omega \\
&\quad \text{where } \overline{\gamma} = \|\psi_N(\omega)\|^{-2}_{2,N,\sigma} \\
&= (\overline{\gamma} - 1)\mathcal{E}_N(\psi_N(\omega), \psi_N(\omega))_\sigma \\
&\quad + \mathcal{E}_N(\psi_N(\omega), \psi_N(\omega))_\sigma - \mathcal{E}_N(\psi_N(\omega), \psi_N(\omega))_\omega \\
&\leq (\overline{\gamma} - 1)\mathcal{E}_N(\psi_N(\omega), \psi_N(\omega))_\sigma
\end{aligned}$$

since for all edges $a(e,\omega) \geq a(e,\sigma)$.

Note that for all $u \in \mathcal{H}_N$, $0 \leq \|u\|^2_{2,N,\omega} - \|u\|^2_{2,N,\sigma} \leq a(e,\omega)(u^2(x_k) + u^2(y_k))$. Then since $\psi_N(\omega)$ is a normalized eigenfunction and the conductances are uniformly elliptic,

$$0 \leq \overline{\gamma} - 1 \leq C(\psi_N^2(x_k,\omega) + \psi_N^2(y_k,\omega)).$$

Then, $\lambda_N(\sigma) - \lambda_N(\omega) \leq CN^{-2}(\psi_N^2(x_k,\omega) + \psi_N^2(y_k,\omega))$. □

Proof of theorem 4. As in the two preceding situations, we will verify conditions (20) and (21) for $\Delta_k = \mathbb{E}(f_N \mid \mathcal{F}_k) - \mathbb{E}(f_N \mid \mathcal{F}_{k-1})$.

$$\begin{aligned}
|\Delta_k| &\leq \mathbb{E}_\sigma(|f_N([\omega,\sigma]_k) - f_N([\omega,\sigma]_{k-1})|) \\
&= \mathbb{E}_\sigma(|\cdots| \,;\, a(e_k,\omega) \geq a(e_k,\sigma)) + \mathbb{E}_\sigma(|\cdots| \,;\, a(e_k,\omega) < a(e_k,\sigma)) \\
&\leq C\mathbb{E}_\sigma(N^2\psi_N^2(e_k,[\omega,\sigma]_{k-1}) + \psi_N^2(x_k,[\omega,\sigma]_k) + \psi_N^2(y_k,[\omega,\sigma]_k)) \\
&\quad + C\mathbb{E}_\sigma(N^2\psi_N^2(e_k,[\omega,\sigma]_k) + \psi_N^2(x_k,[\omega,\sigma]_{k-1}) \\
&\qquad + \psi_N^2(y_k,[\omega,\sigma]_{k-1})) \\
&\leq CN^{2-d}
\end{aligned}$$

since by (32), $|\psi_N(x)| \leq CN^{-d/2}$, for all $x \in Q_N$.

Pursuing the above calculations, and using (32) again, we find that

$$\begin{aligned}
\Delta_k^2 &\leq CN^{4-d}\mathbb{E}_\sigma(\psi_N^2(e_k,[\omega,\sigma]_{k-1}) + \psi_N^2(e_k,[\omega,\sigma]_k)) \\
&\quad + CN^{-d}\mathbb{E}_\sigma(\psi_N^2(x_k,[\omega,\sigma]_k) + \psi_N^2(y_k,[\omega,\sigma]_k) \\
&\qquad + \psi_N^2(x_k,[\omega,\sigma]_{k-1}) + \psi_N^2(y_k,[\omega,\sigma]_{k-1}))
\end{aligned}$$

Let $U_k(\omega) = CN^{4-d}\psi_N^2(e_k,\omega) + CN^{-d}(\psi_N^2(x_k,\omega) + \psi_N^2(y_k,\omega))$.

We see that $\mathbb{E}(\Delta_k^2 \mid \mathcal{F}_{k-1}) \leq \mathbb{E}(U_k \mid \mathcal{F}_{k-1})$ and

$$\begin{aligned}
\sum_k U_k &< CN^{4-d}\sum_k \psi_N^2(e_k,\omega) + CN^{-d}\sum_k(\psi_N^2(x_k,\omega) + \psi_N^2(y_k,\omega)) \\
&= CN^{4-d}\mathcal{E}(\psi_N,\psi_N)_\omega + CN^{-d}\|\psi_N\|^2_{2,N} \\
&\leq CN^{4-d-2} + CN^{-d} \leq CN^{2-d}.
\end{aligned}$$

Hence (20) holds with $B_0 = CN^{2-d}$ and (21) holds with $B_1 = CN^{2-d}$. Since $B = \max\{B_0, eB_1^2\} \leq CN^{2-d}$, we obtain that for all $N \geq 1$ and $t > 0$,

$$\mathbb{P}(|f_N - \mathbb{E}f_N| \geq t) \leq 4\exp\left(-\frac{t}{C}N^{(d-2)/2}\right) \quad \text{and} \quad \mathbb{V}\mathrm{ar}(f_N) \leq CN^{2-d}.$$

□

5 A random walk in a random potential

Let $(a(e); e \in \mathcal{L}^d)$, $d \geq 1$, be a (non-random) sequence of positive conductances and let $V : \mathbb{Z}^d \times \Omega \to [0, \infty[$. Add to the graph a vertex $\ddagger$ and an edge between $\ddagger$ and each vertex $x \in \mathbb{Z}^d$. In the environment ω, its conductance is given by

$$a(x, \ddagger, \omega) = a(x)V(x, \omega).$$

The transition probabilites of the random walk on $\mathbb{Z}^d \cup \{\ddagger\}$ will be denoted by $\breve{p}(x, y)$ to distinguish them from the transition probabilities of the reversible random walk on $\mathbb{Z}^d$ that are given by $p(x, y) = a(x, y)/a(x)$. The former are defined by $\breve{p}(\ddagger, \ddagger) = 1$, $\breve{p}(\ddagger, x) = 0$ and in the other cases, they are given by the conductances of the edges :

$$\breve{p}(x, y) = \frac{p(x, y)}{1 + V(x)}, \quad \text{and} \quad \breve{p}(x, \ddagger) = \frac{V(x)}{1 + V(x)}, \quad x, y \in \mathbb{Z}^d.$$

The survival probability after each step is $(V(x) + 1)^{-1} \stackrel{\text{def}}{=} e^{-\theta(x)}$. Now assume that $(V(x); x \in \mathbb{Z}^d)$ is a sequence of i.i.d. nonnegative random variables on $(\Omega, \mathcal{F}, I\!\!P)$ that are not concentrated on zero.

Let $T = \inf\{k \geq 0; X_k = \ddagger\}$. Then the higher order transition probabilities are given by Feynman-Kac formula

$$\breve{p}(x, y, k) \quad = \quad E_x \left(\prod_{j=0}^{k-1} (V(X_j) + 1)^{-1}; X_k = y, k < T \right)$$

where E_x is the expectation with respect to the reversible random walk on $\mathbb{Z}^d$. Let $\breve{P}_x$ be the induced probability on the paths starting at x.

The Green function is defined by $\breve{G}(x, y) = \sum_{k=0}^{\infty} \breve{p}(x, y, k)$. A short calculation shows that, since V is not concentrated on zero, then for all $x, y \in \mathbb{Z}^d$, $d \geq 2$, $\breve{G}(x, y)$ is a random variable with finite moments of all order.

For a direction $x \in \mathbb{Z}^d$, $x \neq 0$, and $N \geq 1$, let

$$f_N(x, \omega) = -N^{-1} \log \breve{G}(x, Nx, \omega).$$

We now prove a lower bound on the variance. The analogue for first passage percolation is given in [16, (1.13)].

Proposition 4 *If $(\theta(x), x \in \mathbb{Z}^d)$, $d \geq 2$, is a sequence of i.i.d. nonnegative random variables not concentrated on zero and such that $I\!\!E(\theta(0)^2) < \infty$, then for all $x \in \mathbb{Z}^d$, $x \neq 0$, and $N \geq 1$,*

$$\mathbf{V}\mathrm{ar}\, f_N(x) \geq N^{-2}\, \mathbf{V}\mathrm{ar}\, \theta(0).$$

Remark. In the particular case of constant conductances, that is $a(e) = 1$ for all edges $e \in \mathcal{L}^d$, Zerner [28] (see also [24] and [26]) showed that if $I\!\!E\theta(0) < \infty$ then $f_N(x)$ converges $I\!\!P$-a.s. If, moreover, $I\!\!E(\theta(0)^2) < \infty$, and if for $d = 2$,

there is $\underline{\nu}$ such that $\theta \geq \underline{\nu} > 0$, then by [28, Theorem C] there is a constant $C < \infty$ such that for all $x \in \mathbb{Z}^d$, $x \neq 0$, and $N \geq 1$,

$$\mathbf{V}\mathrm{ar}\, f_N(x) \leq C\frac{|x|}{N}.$$

Proof. For $x, y \in \mathbb{Z}^d$, let $\tau_y = \inf\{k \geq 0; X_k = y\}$ and $\tau_x^+ = \inf\{k \geq 1; X_k = x\}$ with the convention that $\inf \emptyset = +\infty$.

Then, by conditioning on the time of last visit to x, we see that for $x \neq y$,

$$\breve{G}(x,y) = \breve{P}_x(\tau_y < T)\breve{G}(y,y) = \breve{G}(x,x)\breve{P}_x(\tau_y < \tau_x^+)\breve{G}(y,y)$$

and $\log \breve{G}(x,y) = \log \breve{G}(x,x) + \log \breve{P}_x(\tau_y < \tau_x^+) + \log \breve{G}(y,y)$. Since these are decreasing functions of V, by the FKG inequality (see [1] for a recent account), they are pairwise positively correlated. Then,

$$\begin{aligned}
\mathbf{V}\mathrm{ar} \log \breve{G}(x,y) &\geq \mathbf{V}\mathrm{ar}(\log \breve{P}_x(\tau_y < \tau_x^+)) \\
&\quad +2\,\mathbf{C}\mathrm{ov}(\log \breve{G}(x,x),\ \log \breve{P}_x(\tau_y < \tau_x^+)) \\
&\quad + 2\,\mathbf{C}\mathrm{ov}(\ \log \breve{P}_x(\tau_y < \tau_x^+), \log \breve{G}(y,y)) \\
&\geq \mathbf{V}\mathrm{ar}(\log \breve{P}_x(\tau_y < \tau_x^+)) \\
&= \mathbf{V}\mathrm{ar}\left(\log \sum_{z \sim x} e^{-\theta(x)} p(x,z)\breve{P}_z(\tau_y < \tau_x^+)\right) \\
&= \mathbf{V}\mathrm{ar}(\theta(x)) + \mathbf{V}\mathrm{ar}\left(\log \sum_{z \sim x} p(x,z)\breve{P}_z(\tau_y < \tau_x^+)\right) \\
&\geq \mathbf{V}\mathrm{ar}(\theta(0)).
\end{aligned}$$

□

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